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Suppression of decoherence in a generalization of the spin-bath model

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Abstract
The works on decoherence due to spin baths usually agree in studying a one-spin system in interaction with a large spin bath. In this paper we generalize those models by analyzing a many-spin system and by studying decoherence or its suppression in function of the relation between the numbers of spins of the system and the bath. This model may help to identify clusters of particles unaffected by decoherence, which, as a consequence, can be used to store quantum information.

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1. Introduction

Decoherence refers to the quantum process that turns a coherent pure state into a decohered mixed state. It is essential in the account of the emergence of classicality from quantum behavior, since it explains how interference vanishes in an extremely short decoherence time. The orthodox explanation of the phenomenon is given by the *environment-induced decoherence* approach (see [1–4]), according to which decoherence is a process resulting from the interaction of an open quantum system and its environment. By studying different physical models, it was proved that the reduced state $\rho_S(t) = Tr_E \rho_{SE}(t)$ of the open system rapidly diagonalizes in a well-defined pointer basis, which identifies the candidates for classical states.

The environment-induced approach has been extensively applied to many areas of physics—such as atomic physics, quantum optics and condensed matter—and has acquired a great importance in quantum computation, where the loss of coherence represents a major difficulty for the implementation of the information processing hardware that takes the advantage of superpositions. In particular, decoherence resulting from the interaction with nuclear spins is the main obstacle to quantum computations in magnetic systems. This fact has led to a growing interest in the study of decoherence due to spin-baths (see [5–14]). By beginning from the seminal paper of Zurek [1], many works have studied the decoherence due to a collection of independent spins. More recently, some papers have directed attention to the interactions between modes within the bath. For instance, by studying a central spin coupled to a spin-bath, Tessieri and Wilkie [6] showed that, whereas in the absence of intra-environmental coupling the decoherence of the central spin is fast and irreversible, strong intra-environmental coupling leads to decoherence suppression. The same model was further analyzed by Dawson *et al* [7], with the purpose of relating decoherence with the pairwise entanglement between individual bath-spins. In turn, Rossini *et al* [10] left behind the assumption that the central spin is coupled isotropically to all the spins of the bath, and considered the case where the spin system interacts with only few spins of the bath.

Our analysis can be framed in the context of the above works; it aims at generalizing the paradigmatic spin-bath model. In fact, most of the works done so far agree in studying a one-spin system in interaction with a large spin bath. The crucial feature of our work is the analysis of a many-spin system, and the study of decoherence or its suppression in function of the relation between the numbers of spins of the system and the bath. This generalized spin-bath model can also be conceived as a partition of a whole closed system into an open many-spin system and its environment. From this perspective, we can study different partitions of the whole system and identify those for which the selected system does not decohere; this might allow us to define clusters of particles that can be used to store q-bits.

In order to develop our analysis, we will rely on the general framework for decoherence introduced in [15], where the split of a closed quantum system into an open subsystem and its environment is just conceived as a way of selecting a particular space of relevant observables of the whole closed system. Since there are many different spaces of relevant observables depending on the observational viewpoint adopted, the same closed system can be decomposed in many different ways: each decomposition represents a decision about which degrees of freedom are relevant and which can be disregarded in each case.

On this basis, the paper is organized as follows. In section 2, the standard spin-bath model is presented from the general framework perspective: this presentation will allow us to consider two different decompositions, which supply the basis for comparing the results obtained for the generalized model in the following sections. In sections 3, 4 and 5, the generalization of the spin-bath model is presented and solved by computer simulations; this task will allow us to compare the results obtained for two different ways of splitting the entire closed system into an open system and its environment. Finally, in section 6 we introduce our concluding remarks.

2. The spin-bath model

The spin-bath model is a very simple model that has been exactly solved in previous papers (see [1]). Here we will recall its main results, obtained from the general framework introduced in [15], in order to compare the analogous results to be obtained in the next sections for the generalized model.

2.1. Presentation of the model

Let us consider a closed system $U = P \cup P_1 \cup \ldots \cup P_N = P \cup \left(\cup_{i=1}^{N} P_i\right)$, where (i) P is a spin-1/2 particle represented in the Hilbert space $\mathcal{H}_P$ and (ii) each P_i is a spin-1/2 particle represented in its Hilbert space $\mathcal{H}_i$. The Hilbert space of the composite system U is, then,

$$\mathcal{H} = \mathcal{H}_P \otimes \left(\bigotimes_{i=1}^{N} \mathcal{H}_i \right). \tag{1}$$

In the particle P, the two eigenstates of the spin operator $S_{P,\vec{v}}$ in the direction $\vec{v}$ are $|\Uparrow\rangle, |\Downarrow\rangle$, such that $S_{P,\vec{v}}|\Uparrow\rangle = \frac{1}{2}|\Uparrow\rangle$ and $S_{P,\vec{v}}|\Downarrow\rangle = -\frac{1}{2}|\Downarrow\rangle$. In each particle P_i, the two eigenstates of the corresponding spin operator $S_{i,\vec{v}}$ in the direction $\vec{v}$ are $|\uparrow_i\rangle, |\downarrow_i\rangle$, such that $S_{i,\vec{v}}|\uparrow_i\rangle = \frac{1}{2}|\uparrow_i\rangle$ and $S_{i,\vec{v}}|\downarrow_i\rangle = -\frac{1}{2}|\downarrow_i\rangle$. Therefore, a pure initial state of U reads

$$|\psi_0\rangle = (a|\Uparrow\rangle + b|\Downarrow\rangle) \otimes \left(\bigotimes_{i=1}^{N}(\alpha_i|\uparrow_i\rangle + \beta_i|\downarrow_i\rangle)\right), \tag{2}$$

where $|a|^2 + |b|^2 = 1$ and $|\alpha_i|^2 + |\beta_i|^2 = 1$. If the self-Hamiltonians H_P of P and H_i of P_i are taken to be zero, and there is no interaction among the P_i, then the total Hamiltonian H of the composite system U is given by the interaction between the particle P and each particle P_i (see [1, 16]):

$$H = \frac{1}{2}(|\Uparrow\rangle\langle\Uparrow| - |\Downarrow\rangle\langle\Downarrow|) \otimes \sum_{i=1}^{N}\left[g_i\,(|\uparrow_i\rangle\langle\uparrow_i| - |\downarrow_i\rangle\langle\downarrow_i|) \otimes \left(\bigotimes_{j\neq i}^{N}\mathbb{I}_j\right)\right], \tag{3}$$

where $\mathbb{I}_j = |\uparrow_j\rangle\langle\uparrow_j| + |\downarrow_j\rangle\langle\downarrow_j|$ is the identity operator on the subspace $\mathcal{H}_j$. Under the action of H, the state $|\psi_0\rangle$ evolves into $|\psi(t)\rangle = a|\Uparrow\rangle|\mathcal{E}_\Uparrow(t)\rangle + b|\Downarrow\rangle|\mathcal{E}_\Downarrow(t)\rangle$, where

$$|\mathcal{E}_\Uparrow(t)\rangle = |\mathcal{E}_\Downarrow(-t)\rangle = \bigotimes_{i=1}^{N}\left(\alpha_i\,\mathrm{e}^{-\mathrm{i}g_i t/2}|\uparrow_i\rangle + \beta_i\,\mathrm{e}^{\mathrm{i}g_i t/2}|\downarrow_i\rangle\right). \tag{4}$$

The space $\mathcal{O}$ of the observables of the composite system U can be obtained as $\mathcal{O} = \mathcal{O}_P \otimes \left(\otimes_{i=1}^{N}\mathcal{O}_i\right)$, where $\mathcal{O}_P$ is the space of the observables of the particle P and $\mathcal{O}_i$ is the space of the observables of the particle P_i. Then, an observable $O \in \mathcal{O} = \mathcal{H}\otimes\mathcal{H}$ can be expressed as

$$O = O_P \otimes \left(\bigotimes_{i=1}^{N} O_i\right), \tag{5}$$

where

$$O_P = s_{\Uparrow\Uparrow}|\Uparrow\rangle\langle\Uparrow| + s_{\Uparrow\Downarrow}|\Uparrow\rangle\langle\Downarrow| + s_{\Downarrow\Uparrow}|\Downarrow\rangle\langle\Uparrow| + s_{\Downarrow\Downarrow}|\Downarrow\rangle\langle\Downarrow| \in \mathcal{O}_P, \tag{6}$$

$$O_i = \epsilon^{(i)}_{\uparrow\uparrow}|\uparrow_i\rangle\langle\uparrow_i| + \epsilon^{(i)}_{\downarrow\downarrow}|\downarrow_i\rangle\langle\downarrow_i| + \epsilon^{(i)}_{\downarrow\uparrow}|\downarrow_i\rangle\langle\uparrow_i| + \epsilon^{(i)}_{\uparrow\downarrow}|\uparrow_i\rangle\langle\downarrow_i| \in \mathcal{O}_i. \tag{7}$$

Since the operators O_P and O_i are Hermitian, the diagonal components $s_{\Uparrow\Uparrow}, s_{\Downarrow\Downarrow}, \epsilon^{(i)}_{\uparrow\uparrow}, \epsilon^{(i)}_{\downarrow\downarrow}$ are real numbers, and the off-diagonal components are complex numbers satisfying $s_{\Uparrow\Downarrow} = s^*_{\Downarrow\Uparrow}, \epsilon^{(i)}_{\uparrow\downarrow} = \epsilon^{(i)*}_{\downarrow\uparrow}$. Then, the expectation value of the observable O in the state $|\psi(t)\rangle$ can be computed as

$$\langle O\rangle_{\psi(t)} = (|a|^2 s_{\Uparrow\Uparrow} + |b|^2 s_{\Downarrow\Downarrow})\,\Gamma_0(t) + 2\,\mathrm{Re}[ab^*\,s_{\Downarrow\Uparrow}\,\Gamma_1(t)], \tag{8}$$

where (see [16])

$$\Gamma_0(t) = \prod_{i=1}^{N}\left[|\alpha_i|^2\epsilon^{(i)}_{\uparrow\uparrow} + |\beta_i|^2\epsilon^{(i)}_{\downarrow\downarrow} + 2\,\mathrm{Re}\left(\alpha_i\,\beta_i^*\epsilon^{(i)}_{\downarrow\uparrow}\mathrm{e}^{\mathrm{i}g_i t}\right)\right], \tag{9}$$

$$\Gamma_1(t) = \prod_{i=1}^{N}\left[|\alpha_i|^2\epsilon^{(i)}_{\uparrow\uparrow}\,\mathrm{e}^{\mathrm{i}g_i t} + |\beta_i|^2\epsilon^{(i)}_{\downarrow\downarrow}\,\mathrm{e}^{-\mathrm{i}g_i t} + 2\,\mathrm{Re}\left(\alpha_i\,\beta_i^*\epsilon^{(i)}_{\downarrow\uparrow}\right)\right]. \tag{10}$$

In contrast to the usual presentations, we will study two different decompositions of the whole closed system U into a relevant part and its environment.

2.2. The spin-bath model: decomposition 1

In the typical presentations of the model, the open system S is the particle P, and the remaining particles P_i play the role of the environment E: $S = P$ and $E = \cup_{i=1}^{N} P_i$. Then, the Hilbert space decomposition for this case is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_P) \otimes \left(\bigotimes_{i=1}^{N} \mathcal{H}_i \right). \tag{11}$$

Therefore, the relevant observables O_R of the closed system U are those corresponding to the particle P, and they are obtained from equations (5)–(7), by making $\epsilon_{\uparrow\uparrow}^{(i)} = \epsilon_{\downarrow\downarrow}^{(i)} = 1$ and $\epsilon_{\uparrow\downarrow}^{(i)} = 0$:

$$O_R = O_S \otimes \mathbb{I}_E = \left(\sum_{s,s'=\Uparrow,\Downarrow} s_{ss'} |s\rangle\langle s'| \right) \otimes \left(\bigotimes_{i=1}^{N} \mathbb{I}_i \right). \tag{12}$$

The expectation value of these observables in the state $|\psi(t)\rangle$ is given by

$$\langle O_R \rangle_{\psi(t)} = |a|^2 s_{\Uparrow\Uparrow} + |b|^2 s_{\Downarrow\Downarrow} + 2\mathrm{Re}[ab^* s_{\Downarrow\Uparrow} r(t)], \tag{13}$$

where

$$r(t) = \langle \mathcal{E}_{\Downarrow}(t) | \mathcal{E}_{\Uparrow}(t) \rangle = \prod_{i=1}^{N} (|\alpha_i|^2 \, \mathrm{e}^{-\mathrm{i}g_i t} + |\beta_i|^2 \, \mathrm{e}^{\mathrm{i}g_i t}) \tag{14}$$

and, then,

$$|r(t)|^2 = \prod_{i=1}^{N} (|\alpha_i|^4 + |\beta_i|^4 + 2|\alpha_i|^2 |\beta_i|^2 \cos 2g_i t). \tag{15}$$

This means that, in equation (8), $\Gamma_0(t) = 1$ and $\Gamma_1(t) = r(t)$.

If we take $|\alpha_i|^2$ and $|\beta_i|^2$ as random numbers in the closed interval [0, 1], then $|r(t)|^2$ is an infinite product of numbers belonging to the open interval (0, 1). As a consequence, $\lim_{N\to\infty} r(t) = 0$. Therefore, it can be expected that, for N finite, $r(t)$ will evolve in time from $r(0) = 1$ to a very small value (see numerical simulations in [1, 16]).

2.3. The spin-bath model: decomposition 2

Although in the usual presentations of the model the open system of interest is P, we can conceive different ways of splitting the whole closed system U into an open system S and its environment E. For instance, we can decide to observe a particular particle P_j of what was previously considered the environment, and to consider the remaining particles as the new environment, in such a way that $S = P_j$ and $E = P \cup \left(\cup_{i=1, i\neq j}^{N} P_i \right)$. The total Hilbert space of the closed composite system U is still given by equation (1), but in this case the corresponding decomposition is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = (\mathcal{H}_j) \otimes \left(\mathcal{H}_P \otimes \left(\bigotimes_{\substack{i=1 \\ i\neq j}}^{N} \mathcal{H}_i \right) \right), \tag{16}$$

and the relevant observables O_R of the closed system U are those corresponding to the particle P_j:

$$O_R = O_S \otimes \mathbb{I}_E = O_{P_j} \otimes \left(\mathbb{I}_P \otimes \left(\bigotimes_{\substack{i=1 \\ i \neq j}}^{N} \mathbb{I}_i \right) \right), \tag{17}$$

where (see equation (7))

$$O_{P_j} = \epsilon_{\uparrow\uparrow}^{(j)} |\uparrow_j\rangle\langle\uparrow_j| + \epsilon_{\downarrow\downarrow}^{(j)} |\downarrow_j\rangle\langle\downarrow_j| + \epsilon_{\downarrow\uparrow}^{(j)} |\downarrow_j\rangle\langle\uparrow_j| + \epsilon_{\uparrow\downarrow}^{(j)} |\uparrow_j\rangle\langle\downarrow_j|. \tag{18}$$

$\mathbb{I}_P$ is the identity operator on the subspace $\mathcal{H}_P$, and the coefficients $\epsilon_{\uparrow\uparrow}^{(j)}, \epsilon_{\downarrow\downarrow}^{(j)}, \epsilon_{\downarrow\uparrow}^{(j)}$ are now generic. The expectation value of the observables O_R in the state $|\psi(t)\rangle$ is given by

$$\langle O_R\rangle_{\psi(t)} = \langle\psi(t)|O_{R_j}|\psi(t)\rangle = |\alpha_j|^2\epsilon_{\uparrow\uparrow}^{(j)} + |\beta_j|^2\epsilon_{\downarrow\downarrow}^{(j)} + 2\mathrm{Re}\big(\alpha_j\beta_j^*\epsilon_{\downarrow\uparrow}^{(j)}\,\mathrm{e}^{ig_jt}\big). \tag{19}$$

Here there is no need of numerical simulations to see that the third term of equation (19) is an oscillating function which, as a consequence, has no limit for $t \to \infty$. This result is not surprising since, in this case, the particle P_j is uncoupled to the particles of its environment.

3. A generalized spin-bath model: presentation of the model

Let us consider a closed system $U = A \cup B$ where

(a) The subsystem A is composed of M spin-1/2 particles A_i, with $i = 1, 2, \ldots, M$, each one of them represented in its Hilbert space $\mathcal{H}_{A_i}$. In each A_i, the two eigenstates of the spin operator $S_{A_i,\vec{v}}$ in direction $\vec{v}$ are $|\Uparrow_i\rangle$ and $|\Downarrow_i\rangle$:

$$S_{A_i,\vec{v}}\,|\Uparrow_i\rangle = \tfrac{1}{2}|\Uparrow_i\rangle, \qquad S_{A_i,\vec{v}}\,|\Downarrow_i\rangle = -\tfrac{1}{2}|\Downarrow_i\rangle. \tag{20}$$

The Hilbert space of A is $\mathcal{H}_A = \bigotimes_{i=1}^{M} \mathcal{H}_{A_i}$. Then, a pure initial state of A reads

$$|\psi_A\rangle = \bigotimes_{i=1}^{M} (a_i\,|\Uparrow_i\rangle + b_i\,|\Downarrow_i\rangle), \qquad \text{with} \quad |a_i|^2 + |b_i|^2 = 1. \tag{21}$$

(b) The subsystem B is composed of N spin-1/2 particles B_k, with $k = 1, 2, \ldots, N$, each one of them represented in its Hilbert space $\mathcal{H}_{B_k}$. In each B_k, the two eigenstates of the spin operator $S_{B_k,\vec{v}}$ in direction $\vec{v}$ are $|\uparrow_k\rangle$ and $|\downarrow_k\rangle$:

$$S_{B_k,\vec{v}}\,|\uparrow_k\rangle = \tfrac{1}{2}\,|\uparrow_k\rangle, \qquad S_{B_k,\vec{v}}\,|\downarrow_k\rangle = -\tfrac{1}{2}\,|\downarrow_k\rangle. \tag{22}$$

The Hilbert space of B is $\mathcal{H}_B = \bigotimes_{k=1}^{N} \mathcal{H}_{B_k}$. Then, a pure initial state of B reads

$$|\psi_B\rangle = \bigotimes_{k=1}^{N} (\alpha_k\,|\uparrow_k\rangle + \beta_k\,|\downarrow_k\rangle), \qquad \text{with} \quad |\alpha_k|^2 + |\beta_k|^2 = 1. \tag{23}$$

The Hilbert space of the composite system $U = A \cup B$ is, then,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B = \left(\bigotimes_{i=1}^{M} \mathcal{H}_{A_i}\right) \otimes \left(\bigotimes_{k=1}^{N} \mathcal{H}_{B_k}\right). \tag{24}$$

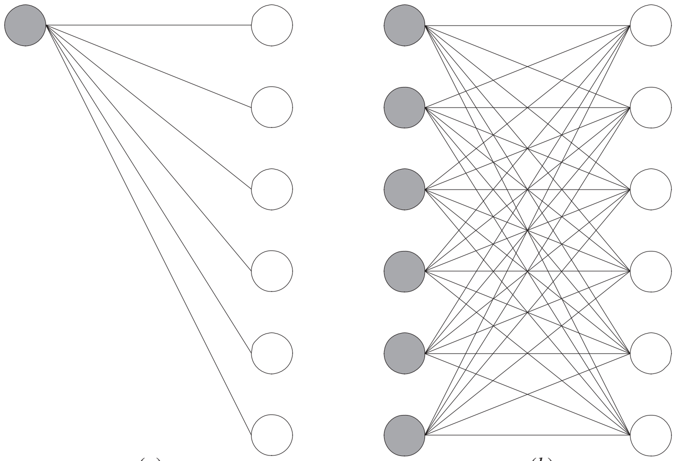

Figure 1. Schema of the interactions among the particles of the open system A (gray circles) and of the open system B (white circles): (*a*) original spin-bath model ($M = 1$), and (*b*) generalized spin-bath model ($M \neq 1$).

Therefore, from equations (21) and (23), a pure initial state of U reads

$$|\psi_0\rangle = |\psi_A\rangle \otimes |\psi_B\rangle = \left(\bigotimes_{i=1}^{M} (a_i\,|{\Uparrow_i}\rangle + b_i\,|{\Downarrow_i}\rangle)\right) \otimes \left(\bigotimes_{k=1}^{N} (\alpha_k\,|{\uparrow_k}\rangle + \beta_k\,|{\downarrow_k}\rangle)\right). \tag{25}$$

As in the original spin-bath model, the self-Hamiltonians H_{A_i} and H_{B_k} are taken to be zero. In turn, there is no interaction among the particles A_i nor among the particles B_k. As a consequence, the total Hamiltonian H of the composite system U is given by

$$H = H_A \otimes H_B = \left(\sum_{i=1}^{M}\left[\frac{1}{2}\,(|{\Uparrow_i}\rangle\,\langle{\Uparrow_i}| - |{\Downarrow_i}\rangle\,\langle{\Downarrow_i}|) \otimes \left(\bigotimes_{j\neq i}^{M} \mathbb{I}_{A_j}\right)\right]\right) \otimes \left(\sum_{k=1}^{N}\left[g_k\,(|{\uparrow_k}\rangle\,\langle{\uparrow_k}| - |{\downarrow_k}\rangle\,\langle{\downarrow_k}|) \otimes \left(\bigotimes_{l\neq k}^{N} \mathbb{I}_{B_l}\right)\right]\right), \tag{26}$$

where $\mathbb{I}_{A_j} = |{\Uparrow_j}\rangle\langle{\Uparrow_j}| + |{\Downarrow_j}\rangle\langle{\Downarrow_j}|$ is the identity on the subspace $\mathcal{H}_{A_j}$ and $\mathbb{I}_{B_l} = |{\uparrow_l}\rangle\,\langle{\uparrow_l}| + |{\downarrow_l}\rangle\,\langle{\downarrow_l}|$ is the identity on the subspace $\mathcal{H}_{B_l}$. Let us note that equation (3) of the original model is the particular case of equation (26) for $M = 1$. This Hamiltonian describes a situation where the particles of A do not interact with each other, the same holds for the particles of B, but each particle of A interacts with all the particles of B and vice versa, as shown in figure 1.

In equation (26), H is written in its diagonal form; then, the energy eigenvectors are

$$\begin{aligned}
&|{\Uparrow_1}\rangle \cdots |{\Uparrow_i}\rangle \cdots |{\Uparrow_{M-1}}\rangle\,|{\Uparrow_M}\rangle\,|{\uparrow_1}\rangle \cdots |{\uparrow_k}\rangle \cdots |{\uparrow_{N-1}}\rangle\,|{\uparrow_N}\rangle,\\
&|{\Uparrow_1}\rangle \cdots |{\Uparrow_i}\rangle \cdots |{\Uparrow_{M-1}}\rangle\,|{\Uparrow_M}\rangle\,|{\uparrow_1}\rangle \cdots |{\uparrow_k}\rangle \cdots |{\uparrow_{N-1}}\rangle\,|{\downarrow_N}\rangle,\\
&\cdots\\
&|{\Downarrow_1}\rangle \cdots |{\Downarrow_i}\rangle \cdots |{\Downarrow_{M-1}}\rangle\,|{\Downarrow_M}\rangle\,|{\downarrow_1}\rangle \cdots |{\downarrow_k}\rangle \cdots |{\downarrow_{N-1}}\rangle\,|{\downarrow_N}\rangle.
\end{aligned} \tag{27}$$

In turn, the eigenvectors of H_A form a basis of $\mathcal{H}_A$. In order to simplify the expressions, we will introduce a particular arrangement into the set of those vectors, by calling them $|\mathcal{A}_i\rangle$: the set $\{|\mathcal{A}_i\rangle\}$ is an eigenbasis of H_A with 2^M elements. The $|\mathcal{A}_i\rangle$ will be ordered in terms of the number $l \in \mathbb{N}_0$ of particles of A having spin $|\Downarrow\rangle$. Then, we have the following.

- $l = 0$ corresponds to the unique state with all the particles with spin $|\Uparrow\rangle$:

$$|\mathcal{A}_1\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Uparrow\rangle \Longrightarrow H_A\,|\mathcal{A}_1\rangle = \frac{M}{2}\,|\mathcal{A}_1\rangle\,. \tag{28}$$

- $l = 1$ corresponds to the M states with only one particle with spin $|\Downarrow\rangle$. Since the order of the eigenvectors with the same eigenvalue will be irrelevant for the computations, we will order these states in an arbitrary way:

$$|\mathcal{A}_j\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Downarrow, \Uparrow, \ldots, \Uparrow, \Uparrow\rangle \Longrightarrow H_A|\mathcal{A}_j\rangle = \frac{M-2}{2}|\mathcal{A}_j\rangle,$$
$$\text{with} \qquad j = 2, 3, \ldots, M+1. \tag{29}$$

- $l = 2$ corresponds to the $\frac{(M-1)M}{2}$ states with two particles with spin $|\Downarrow\rangle$. Again, we will order these states in an arbitrary way:

$$|\mathcal{A}_j\rangle = |\Uparrow, \Uparrow, , \ldots, \Uparrow, \Downarrow, \Uparrow, , \ldots, \Uparrow, \Downarrow, \Uparrow, \ldots, \Uparrow, \Uparrow\rangle \Longrightarrow H_A|\mathcal{A}_j\rangle = \frac{M-4}{2}|\mathcal{A}_j\rangle,$$
$$\text{with} \qquad j = M+2, M+3, \ldots, M+1+\frac{(M-1)\,M}{2}. \tag{30}$$

- For the remaining values of l, the procedure is analogous.

  Consequently, we have

$$\begin{aligned}
&1 \text{ eigenvector with eigenvalue } \frac{M}{2},\\
&M \text{ eigenvectors with eigenvalue } \frac{M-2}{2},\\
&\vdots\\
&\frac{M!}{(M-l)!l!} \text{ eigenvectors with eigenvalue } \frac{M-2l}{2},
\end{aligned} \tag{31}$$

with $l = 0, 1, \ldots, M$. Then, it is clear that H_A is degenerate: it has 2^M eigenvectors but only M different eigenvalues. Therefore, a generic state $|\mathcal{A}\rangle$ of the system A can be written in the basis $\{|\mathcal{A}_i\rangle\}$ as

$$|\mathcal{A}\rangle = \sum_{i=1}^{2^M} C_i|\mathcal{A}_i\rangle \in \mathcal{H}_A, \qquad \text{with} \quad \sum_{i=1}^{2^M} |C_i|^2 = 1. \tag{32}$$

By introducing equation (32) into equation (25), a pure initial state of the composite system $U = A \cup B$ reads

$$|\psi_0\rangle = \left(\sum_{i=1}^{2^M} C_i|\mathcal{A}_i\rangle\right) \otimes \left(\bigotimes_{k=1}^{N} (\alpha_k|\uparrow_k\rangle + \beta_k|\downarrow_k\rangle)\right). \tag{33}$$

If we group the degrees of freedom of B in a single ket $|\mathcal{B}(0)\rangle$, $|\psi_0\rangle$ results

$$|\psi_0\rangle = \sum_{i=1}^{2^M} C_i|\mathcal{A}_i\rangle \otimes |\mathcal{B}(0)\rangle. \tag{34}$$

The time evolution of $|\psi(t)\rangle$ is ruled by the time-evolution operator $\mathcal{U}(t) = \mathrm{e}^{-\mathrm{i}Ht} = \mathrm{e}^{-\mathrm{i}(H_A \otimes H_B)t}$:

$$|\psi(t)\rangle = \mathcal{U}(t)|\psi_0\rangle = \sum_{i=1}^{2^M} C_i\, \mathrm{e}^{-\mathrm{i}(H_A \otimes H_B)t}\, |\mathcal{A}_i\rangle \otimes |\mathcal{B}(0)\rangle$$
$$= \sum_{i=1}^{2^M} C_i\, \mathrm{e}^{-\mathrm{i}H_A t}\, |\mathcal{A}_i\rangle \otimes \mathrm{e}^{-\mathrm{i}H_B t}|\mathcal{B}(0)\rangle. \tag{35}$$

If we use Λ_k to denote the eigenvalue of H_A corresponding to the eigenvector $|\mathcal{A}_k\rangle$, then

$$|\psi(t)\rangle = \sum_{i=1}^{2^M} C_i\, |\mathcal{A}_i\rangle \otimes \mathrm{e}^{-\mathrm{i}\Lambda_i H_B t}\, |\mathcal{B}(0)\rangle = \sum_{i=1}^{2^M} C_i\, |\mathcal{A}_i\rangle \otimes |\mathcal{B}(t)\rangle\,, \tag{36}$$

where (see equation (26))

$$|\mathcal{B}(t)\rangle = \mathrm{e}^{-\mathrm{i}\Lambda_i H_B t}\, |\mathcal{B}(0)\rangle = \exp\left[-\mathrm{i}\Lambda_k \sum_{j=1}^{N} g_j (|\uparrow_j\rangle\langle\uparrow_j| - |\downarrow_j\rangle\langle\downarrow_j|)t\right] |\mathcal{B}(0)\rangle\,. \tag{37}$$

Since the number of the eigenstates of H_A with the same eigenvalue is given by equation (31), the terms of $|\psi(t)\rangle$ can be arranged as

$$|\psi(t)\rangle = (C_1|\mathcal{A}_1\rangle\, |\mathcal{B}_0(t)\rangle) + \left(\sum_{\lambda=1}^{M+1} C_\lambda |\mathcal{A}_\lambda\rangle\, |\mathcal{B}_1(t)\rangle\right) + \left(\sum_{\lambda=M+2}^{M+1+\frac{(M-1)M}{2}} C_\lambda\, |\mathcal{A}_\lambda\rangle\, |\mathcal{B}_2(t)\rangle\right) + \cdots +$$
$$+ \left(\sum_{\lambda=1+\sum_{p=0}^{l-1}\binom{M}{p}}^{\sum_{p=0}^{l}\binom{M}{p}} C_\lambda\, |\mathcal{A}_\lambda\rangle\, |\mathcal{B}_l(t)\rangle\right) + \cdots + (C_{2^M}\, |\mathcal{A}_{2^M}\rangle\, |\mathcal{B}_M(t)\rangle)\,, \tag{38}$$

where

$$|\mathcal{B}_l(t)\rangle = \bigotimes_{k=1}^{N} \left(\alpha_k\, \mathrm{e}^{\mathrm{i}\frac{(2l-M)}{2} g_k t}\, |\uparrow_k\rangle + \beta_k\, \mathrm{e}^{-\mathrm{i}\frac{(2l-M)}{2} g_k t}\, |\downarrow_k\rangle\right). \tag{39}$$

If we compare equation (39) with equation (4), we can see that $|\mathcal{E}_\Uparrow(t)\rangle$ and $|\mathcal{E}_\Downarrow(t)\rangle$ are the particular cases of $|\mathcal{B}_l(t)\rangle$ for$M = 1$ and, then, $l = 0, 1$. Let us recall that l is the number of particles of the system A having spin $|\Downarrow\rangle$. Then, with $M = 1$ and $l = 0$, $|\mathcal{B}_l(t)\rangle = |\mathcal{E}_\Uparrow(t)\rangle$, and with $M = 1$ and $l = 1$, $|\mathcal{B}_l(t)\rangle = |\mathcal{E}_\Downarrow(t)\rangle$.

If we define the function

$$f(l) = \begin{Bmatrix} \sum_{p=0}^{l}\binom{M}{p} & \text{if} \quad l = 0, 1, \ldots, M \\ 0 & \text{otherwise} \end{Bmatrix}, \tag{40}$$

then equation (38) can be rewritten as

$$|\psi(t)\rangle = \sum_{l=0}^{M} \sum_{\lambda=f(l-1)+1}^{f(l)} C_\lambda\, |\mathcal{A}_\lambda\rangle\, |\mathcal{B}_l(t)\rangle\,, \tag{41}$$

and the state operator $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$ reads

$$\rho(t) = \sum_{l,l'=0}^{M} \sum_{\substack{\lambda=f(l-1)+1 \\ \lambda'=f(l'-1)+1}}^{\substack{f(l) \\ f(l')}} C_\lambda C^*_{\lambda'}\, |\mathcal{A}_\lambda\rangle\, |\mathcal{B}_l(t)\rangle\, \langle\mathcal{B}_{l'}(t)|\, \langle\mathcal{A}_{\lambda'}|\,. \tag{42}$$

An observable $O \in \mathcal{O} = \mathcal{H} \otimes \mathcal{H}$ of the closed system $U = A \cup B$ can be expressed as

$$O = \left(\sum_{\lambda,\lambda'=0}^{2^M} s_{\lambda,\lambda'} |\mathcal{A}_\lambda\rangle\langle\mathcal{A}_{\lambda'}| \right) \otimes \left(\bigotimes_{i=1}^{N} \left(\epsilon_{\uparrow\uparrow}^{(i)} |\uparrow_i\rangle\langle\uparrow_i| + \epsilon_{\uparrow\downarrow}^{(i)} |\uparrow_i\rangle\langle\downarrow_i| + \epsilon_{\downarrow\uparrow}^{(i)} |\downarrow_i\rangle\langle\uparrow_i| + \epsilon_{\downarrow\downarrow}^{(i)} |\downarrow_i\rangle\langle\downarrow_i| \right) \right). \quad (43)$$

Let us note that equation (5) (a generic observable in the original spin-bath model) is a particular case of this equation (43), with only four terms in the first factor. Analogously to that case, the diagonal components $s_{\lambda,\lambda}, \epsilon_{\uparrow\uparrow}^{(i)}, \epsilon_{\downarrow\downarrow}^{(i)}$ are real numbers, and the off-diagonal components are complex numbers satisfying $s_{\lambda,\lambda'} = s_{\lambda',\lambda}^{*}, \epsilon_{\uparrow\downarrow}^{(i)} = \epsilon_{\downarrow\uparrow}^{(i)*}$. Then, the expectation value of the observable O in the state $\rho(t)$ of equation (42) can be computed as

$$\langle O\rangle_{\rho(t)} = Tr\,(O\rho(t)) = \sum_{l,l'=0}^{M} \sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(l'-1)+1}}^{\substack{f(l)\\ f(l')}} B_{\lambda,\lambda'} T_{l,l'}(t), \quad (44)$$

where

$$T_{l,l'}(t) = \prod_{j=1}^{N} \left[|\alpha_j|^2 \epsilon_{\uparrow\uparrow}^{(j)} \,\mathrm{e}^{\mathrm{i}(g_{j,l}-g_{j,l'})\frac{t}{2}} + |\beta_j|^2 \epsilon_{\downarrow\downarrow}^{(j)} \,\mathrm{e}^{-\mathrm{i}(g_{j,l}-g_{j,l'})\frac{t}{2}} + 2\,\mathrm{Re}\big(\alpha_j \beta_j^{*} \epsilon_{\downarrow\uparrow}^{(j)} \,\mathrm{e}^{\mathrm{i}(g_{j,l}+g_{j,l'})\frac{t}{2}}\big) \right] \quad (45)$$

and

$$g_{j,l} = (2l - M)\, g_j, \qquad B_{\lambda,\lambda'} = C_\lambda C_{\lambda'}^{*} s_{\lambda',\lambda}. \quad (46)$$

Since the exponents in equation (45) are of the form $g_{j,l} \pm g_{j,l'}$, in some cases they are zero. So, we can write

$$\langle O\rangle_{\rho(t)} = \sum_{l=0}^{M} \sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(l-1)+1}}^{f(l)} B_{\lambda,\lambda'} T_{l,l}(t) + \sum_{l=0}^{\tilde{M}} \sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(M-l-1)+1}}^{\substack{f(l)\\ f(M-l)}} B_{\lambda,\lambda'} 2\,\mathrm{Re}(T_{l,M-l}(t)) + \sum_{\substack{l,l'=0\\ l\neq l'\\ l'\neq M-l}}^{M} \sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(l-1)+1}}^{\substack{f(l)\\ f(l')}} B_{\lambda,\lambda'} T_{l,l'}(t), \quad (47)$$

where

$$\tilde{M} = \begin{Bmatrix} \frac{M-2}{2} & \text{if} \quad M \text{ is even} \\ \frac{M-1}{2} & \text{if} \quad M \text{ is odd} \end{Bmatrix}, \quad (48)$$

$$T_{l,l}(t) = \prod_{j=1}^{N} \left[|\alpha_j|^2 \epsilon_{\uparrow\uparrow}^{(j)} + |\beta_j|^2 \epsilon_{\downarrow\downarrow}^{(j)} + 2\,\mathrm{Re}\big(\alpha_j \beta_j^{*} \epsilon_{\downarrow\uparrow}^{(j)} \,\mathrm{e}^{i g_{j,l} t}\big) \right], \quad (49)$$

$$T_{l,M-l}(t) = \prod_{j=1}^{N} \left[|\alpha_j|^2 \epsilon_{\uparrow\uparrow}^{(j)} \,\mathrm{e}^{i g_{j,l} t} + |\beta_j|^2 \epsilon_{\downarrow\downarrow}^{(j)} \,\mathrm{e}^{-i g_{j,l} t} + 2\,\mathrm{Re}\big(\alpha_j \beta_j^{*} \epsilon_{\downarrow\uparrow}^{(j)}\big) \right]. \quad (50)$$

Let us note that equations (49) and (50) are analogous to equations (9) and (10) for $\Gamma_0(t)$ and $\Gamma_1(t)$, respectively, in the original model, with $g_{j,l} = (2l - M)g_j$ instead of g_j. In particular, when $M = 1$ and, so, $l = 0, 1$, then $T_{l,l}(t) = \Gamma_0(t)$ and $T_{l,M-l}(t) = \Gamma_1(t)$.

As in the case of the original spin-bath model, here we will consider different meaningful ways of selecting the relevant observables.

4. Generalized spin-bath model: decomposition 1

4.1. Selecting the relevant observables

In this case, A is the open system S and B is the environment E. This is a generalization of decomposition 1 in the original spin-bath model. The only difference with respect to that case is that here the system S is composed of $M \geqslant 1$ particles instead of only one. Then, the decomposition for this case is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = \left(\bigotimes_{i=1}^{M} \mathcal{H}_{A_i} \right) \otimes \left(\bigotimes_{k=1}^{N} \mathcal{H}_{B_k} \right). \tag{51}$$

Therefore, the relevant observables O_R of the closed system U are those corresponding to A, and they are obtained from equation (43) by making $\epsilon_{\uparrow\uparrow}^{(i)} = \epsilon_{\downarrow\downarrow}^{(i)} = 1, \epsilon_{\uparrow\downarrow}^{(i)} = 0$ (compare with equation (12) in the original spin-bath model):

$$O_R = O_S \otimes \mathbb{I}_E = \left(\sum_{\lambda,\lambda'=0}^{2^M} s_{\lambda,\lambda'} |\mathcal{A}_\lambda\rangle\langle\mathcal{A}_{\lambda'}| \right) \otimes \left(\bigotimes_{i=1}^{N} \mathbb{I}_i \right). \tag{52}$$

With this condition, the expectation values of these observables are given by equation (47), with

$$T_{l,l}(t) = \prod_{j=1}^{N} (|\alpha_j|^2 + |\beta_j|^2) = 1, \tag{53}$$

$$T_{l,M-l}(t) = \prod_{j=1}^{N} (|\alpha_j|^2 \,\mathrm{e}^{\mathrm{i}g_{j,l}t} + |\beta_j|^2 \,\mathrm{e}^{-\mathrm{i}g_{j,l}t}), \tag{54}$$

$$T_{l,l'}(t) = \prod_{j=1}^{N} (|\alpha_j|^2 \,\mathrm{e}^{\mathrm{i}(g_{j,l}-g_{j,l'})\frac{t}{2}} + |\beta_j|^2 \,\mathrm{e}^{-\mathrm{i}(g_{j,l}-g_{j,l'})\frac{t}{2}}). \tag{55}$$

If we define the functions $R_l(t) = |T_{l,M-l}(t)|^2$ and $R_{ll'}(t) = |T_{l,l'}(t)|^2$, they result

$$R_l(t) = \prod_{j=1}^{N} (|\alpha_j|^4 + |\beta_j|^4 + 2|\alpha_j|^2|\beta_j|^2 \cos(2(2l - M)g_j t)), \tag{56}$$

$$R_{ll'}(t) = \prod_{j=1}^{N} (|\alpha_j|^4 + |\beta_j|^4 + 2|\alpha_j|^2|\beta_j|^2 \cos(2(l - l')g_j t)). \tag{57}$$

We can see that $|r(t)|^2$ of equation (15) in the original model is the particular case of $R_l(t)$ for $M = 1$.

4.2. Computing the behavior of the relevant expectation values

The expectation value given by equation (47) has three terms, $\langle O_R\rangle_{\rho(t)} = \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}$, which can be analyzed separately.

- From equation (53), the first term reads

$$\Sigma^{(1)} = \sum_{l=0}^{M} \sum_{\lambda,\lambda'=f(l-1)+1}^{f(l)} B_{\lambda,\lambda'} = \sum_{l=0}^{M} \sum_{\lambda,\lambda'=f(l-1)+1}^{f(l)} C_\lambda C^*_{\lambda'} s_{\lambda',\lambda} \neq \Sigma^{(1)}(t). \quad (58)$$

It is clear that this first term does not evolve with time.

- The time dependence of the second term is given by $T_{l,M-l}(t)$:

$$\Sigma^{(2)}(t) = \sum_{l=0}^{\tilde{M}} \sum_{\substack{\lambda=f(l-1)+1 \\ \lambda'=f(M-l-1)+1}}^{\substack{f(l) \\ f(M-l)}} B_{\lambda,\lambda'} 2\,\mathrm{Re}(T_{l,M-l}(t)). \quad (59)$$

Then, in order to obtain the limit of this term, we have to compute the limit of $R_l(t) = |T_{l,M-l}(t)|^2$ of equation (56). As in the case of the original spin-bath model, here we take $|\alpha_j|^2$ and $|\beta_{ji}|^2$ as random numbers in the closed interval [0, 1], such that $|\alpha_j|^2 + |\beta_j|^2 = 1$. Then

$$\max_t(|\alpha_j|^4 + |\beta_j|^4 + 2|\alpha_j|^2|\beta_j|^2\cos(2(2l-M)g_j t)) = 1, \quad (60)$$

$$\min_t(|\alpha_j|^4 + |\beta_j|^4 + 2|\alpha_j|^2|\beta_j|^2\cos(2(2l-M)g_j t)) = (2|\alpha_j|^2 - 1)^2. \quad (61)$$

Therefore, $[|\alpha_j|^4 + |\beta_j|^4 + 2|\alpha_j|^2|\beta_j|^2\cos(2(2l-M)g_j t)]$ is a random number which, if $t \neq 0$, fluctuates between 1 and $(2|\alpha_j|^2 - 1)^2$. Again, when the environment has many particles (that is, when $N \to \infty$), the statistical value of the cases $|\alpha_j|^2 = 1$, $|\beta_j|^2 = 1$, $|\alpha_j|^2 = 0$ and $|\beta_j|^2 = 0$ tends to zero. In this situation, equation (56) for $R_l(t)$ is an infinite product of numbers belonging to the open interval (0, 1). As a consequence, when $N \to \infty$, $R_l(t) \to 0$.

- The time dependence of the third term is given by $T_{l,l'}(t)$:

$$\Sigma^{(3)}(t) = \sum_{\substack{l,l'=0 \\ l\neq l' \\ l'\neq M-l}}^{M} \sum_{\substack{\lambda=f(l-1)+1 \\ \lambda'=f(l'-1)+1}}^{\substack{f(l) \\ f(l')}} B_{\lambda,\lambda'} T_{l,l'}(t), \quad (62)$$

with the restrictions on l and l': $l \neq l'$ and $l' \neq M - l$. As in the second term, we have to compute the limit of $R_{ll'}(t) = |T_{l,l'}(t)|^2$ of equation (57) and, on the basis of an analogous argument, the result is the same as above: when $N \to \infty$, $R_{ll'}(t) \to 0$.

If now we want to evaluate the limit of $\langle O_R \rangle_{\rho(t)}$ for $t \to \infty$, we have to compute the limits of the second and the third terms (since the first term, as we have seen, is time independent). Here we have to distinguish three cases: $M \ll N$, $M \gg N$ and $M \simeq N$.

Case (a): $M \ll N$. This case is similar to decomposition 1 in the original spin-bath model, since in both cases $M \ll N$: the only difference is that in the original model $M = 1$ whereas here $M \geqslant 1$.

In fact, we have seen that $T_{l,M-l}(t)$ is analogous to $\Gamma_1(t)$ in the original model. Moreover, $T_{l,l'}(t)$ has the same functional form as $\Gamma_1(t)$. In paper [16] it is shown that $\Gamma_1(t)$ approaches zero for $t \to \infty$. This means that we can infer that $T_{l,M-l}(t)$ and $T_{l,l'}(t)$ also approach zero for $t \to \infty$. On the other hand, the terms $\Sigma^{(2)}(t)$ and $\Sigma^{(3)}(t)$ are sums of less than M terms involving $T_{l,M-l}(t)$ and $T_{l,l'}(t)$. As a consequence, since in this case M is a small number, the sum of a small number of terms approaching zero for $t \to \infty$ also approaches zero: $\lim_{t\to\infty} \Sigma^{(2)}(t) = 0$ and $\lim_{t\to\infty} \Sigma^{(3)}(t) = 0$. Therefore,

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho(t)} = \lim_{t\to\infty}[\Sigma^{(1)}(t) + \Sigma^{(2)}(t) + \Sigma^{(3)}(t)] = \Sigma^{(1)}(t). \tag{63}$$

In other words,

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho(t)} = \sum_{l=0}^{M}\sum_{\lambda,\lambda'=f(l-1)+1}^{f(l)} B_{\lambda,\lambda'} = \sum_{l=0}^{M}\sum_{\lambda,\lambda'=f(l-1)+1}^{f(l)} C_\lambda C^*_{\lambda'} s_{\lambda',\lambda} = \langle O_R\rangle_{\rho_*}, \tag{64}$$

where ρ_* is the final diagonal state of U. This result can also be expressed in terms of the reduced density operator ρ_A of the system A as

$$\lim_{t\to\infty}\langle O_R\rangle_{\rho(t)} = \langle O_R\rangle_{\rho_*} = \lim_{t\to\infty}\langle O_A\rangle_{\rho_A(t)} = \langle O_A\rangle_{\rho_{A*}}. \tag{65}$$

In the eigenbasis of the Hamiltonian H_A of A, the final reduced density operator ρ_{A*} is expressed by a $2^M \times 2^M$ matrix:

$$\rho_{A*} = \begin{pmatrix} \rho_{l=0} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \rho_{l=1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \rho_{l=2} & 0 & \cdots & 0 \\ 0 & 0 & 0 & \rho_{l=3} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \rho_{l=M} \end{pmatrix}, \tag{66}$$

where $\rho_{l=0} = |C_1|^2$ and each ρ_l is a matrix of dimension $\frac{M!}{(M-l)!l!} \times \frac{M!}{(M-l)!l!}$. This result might seem insufficient for decoherence because, since the ρ_l are matrices, ρ_{A*} seems to be non-completely diagonal in the eigenbasis of the Hamiltonian H_A. However, we have to recall that all the states $|\mathcal{A}_i\rangle$ with same l are degenerate eigenvectors corresponding to the same eigenvalue of H_A; then, the basis that diagonalizes ρ_{A*} (i.e. that diagonalizes all the matrices ρ_l) is an eigenbasis of H_A. Summing up, the system $S = A$ of M particles in interaction with its environment $E = B$ of $N \gg M$ particles decoheres in the eigenbasis of ρ_{A*}, which is also an eigenbasis of H_A.

If we want to compute the time behavior of $\langle O_R\rangle_{\rho(t)}$, we have to consider that $\Sigma^{(1)}$ is a sum of terms of the form $(B_{\lambda,\lambda'}|\alpha_j|^2 + B_{\lambda,\lambda'}|\beta_j|^2)$, that is, terms of the expectation value coming from the diagonal part of $\rho(t)$ in the basis of the Hamiltonian H. Therefore, if there is decoherence, the sum $\Sigma^{nd}(t) = \Sigma^{(2)} + \Sigma^{(3)}$, involving the terms of $\langle O_R\rangle_{\rho(t)}$ coming from the non-diagonal part of $\rho(t)$, has to approach zero for $t \to \infty$.

In order to show an example of the time behavior of $\langle O_R\rangle_{\rho(t)}$, numerical simulations for $\Sigma^{nd}(t)$ have been performed, with the following features.

(i) $s_{\lambda',\lambda} = 1$ (see equation (52)).

(ii) The initial condition for $S = A$ is selected as (see equation (32))

$$|\mathcal{A}\rangle = \frac{1}{\sqrt{2^M}}\sum_{i=1}^{2^M}|\mathcal{A}_i\rangle \Longrightarrow \forall\lambda, \quad C_\lambda = C^*_\lambda = \frac{1}{\sqrt{2^M}} \Longrightarrow C_\lambda C^*_{\lambda'} = \frac{1}{2^M}. \tag{67}$$

Then, from (i) and (ii), $B_{\lambda,\lambda'} = 2^{-M}$ (see equation (46)).

(iii) $|\alpha_i|^2$ is generated by a random-number generator in the interval [0, 1] and $|\beta_i|^2$ is obtained as $|\beta_i|^2 = 1 - |\alpha_i|^2$.

(iv) $g_i = 400\,$Hz: as explained above, the coupling constant in typical models of spin interaction.

(v) As in the original model, the time interval $[0, t_0]$ was partitioned into intervals $\Delta t = t_0/200$, and the function $\Sigma^{nd}(t)$ was computed at times $t_k = k\Delta t$, with $k = 0, 1, \ldots, 200$.

(vi) $N = 10^3$, and $M = 1$ and $M = 10$.

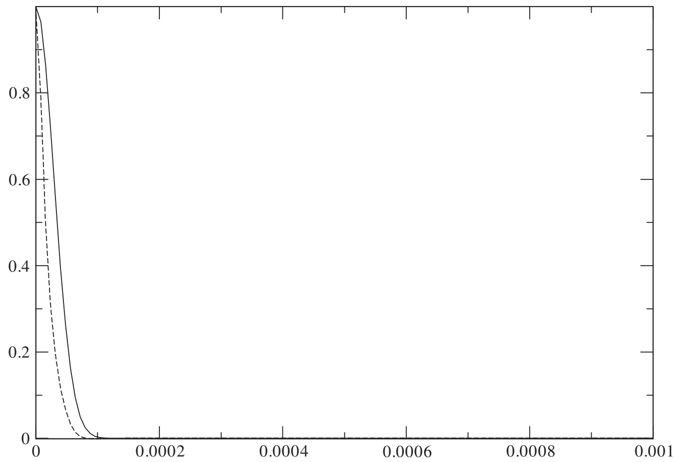

Figure 2. Evolution of $\Sigma^{nd}(t)$ for $N = 10^3$, and $M = 1$ (solid line) and $M = 10$ (dashed line), with $t_0 = 10^{-3}$s.

Figure 2 shows the time evolution of $\Sigma^{nd}(t)$.

This result shows that, as expected, a small open system $S = A$ of M particles decoheres in interaction with a large environment $E = B$ of $N \gg M$ particles.

Case (b): $M \gg N$. In this case, where the open system $S = A$ has much more particles than the environment $E = B$, the argument of case (a) cannot be applied: since now $\Sigma^{(2)}(t)$ and $\Sigma^{(3)}(t)$ are no longer sums over a small number of terms, the fact that each term approaches zero does not guarantee that the sums also approach zero. In particular, if $N = 1$, then (see equation (55))

$$T_{l,l'}(t) = |\alpha_1|^2\, \mathrm{e}^{\mathrm{i}(g_{1,l}-g_{1,l'})\frac{t}{2}} + |\beta_1|^2\, \mathrm{e}^{-\mathrm{i}(g_{1,l}-g_{1,l'})\frac{t}{2}} \qquad (68)$$

which clearly has no limit for $t \to \infty$. Nevertheless, it might happen that, with high N but M much higher than N, each term of the sums approaches zero. So, in order to know the time behavior of $\langle O_R\rangle_{\rho(t)}$, numerical simulations for $\Sigma^{nd}(t)$ have been performed, with the same features as in the previous case, with the exception of condition (vi), which was taken as

(vi) $M = 10^3$, and $N = 10$ and $N = 100$.

Figure 3 shows the time evolution of $\Sigma^{nd}(t)$ in this case.

This result is also what may be expected: when the open system $S = A$ of M particles is larger that the environment $E = B$ of $N \ll M$ particles, S does not decohere.

Case (c): $M \simeq N$. In this case, where the numbers of particles of the open system $S = A$ and of the environment $E = B$ do not differ in more than one order of magnitude, the time behavior of $\langle O_R\rangle_{\rho(t)}$ cannot be inferred from the equations. Numerical simulations have been performed, with the same features as in case (b), with the exception of condition (vi), which was taken as

(vi) $N = 10^3$, and $M = 10^2$ and $M = 10^3$.

Figure 4 shows the time evolution of $\Sigma^{nd}(t)$.

Again, if the environment $E = B$ of N particles is not large enough when compared with the open system $S = A$ of M particles, S does not decohere. Let us note that, for $N = 10^3$,

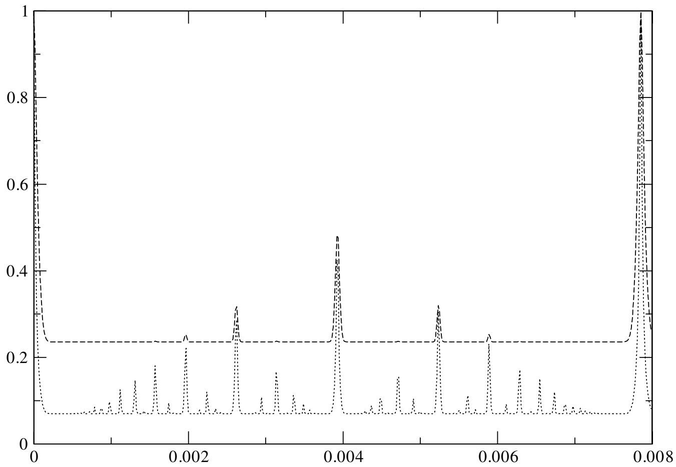

Figure 3. Evolution of $\Sigma^{nd}(t)$ for $M = 10^3$, and $N = 10$ (dashed line) and $N = 100$ (dotted line), with $t_0 = 10^{-3}$s.

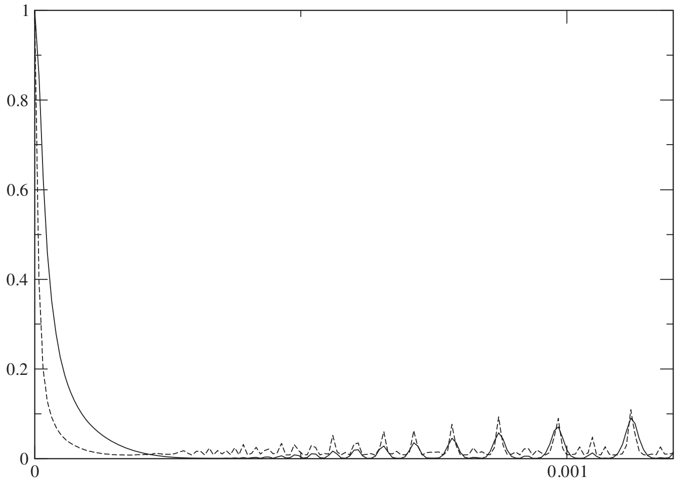

Figure 4. Evolution of $\Sigma^{nd}(t)$ for $N = 10^3$, and $M = 10^2$ (dashed line) and $M = 10^3$ (solid line), with $t_0 = 12.10^{-4}$s.

the system $S = A$ with $M = 10^2$ does not decohere (figure 4), whereas it does decohere with $M = 10$ (figure 2). This shows that, in the case of this decomposition, $M \ll N$ means that N is at least two orders of magnitude higher than M.

Summarizing results

Up to now, in this decomposition 1 all the arguments were directed to know whether the system A of M particles decoheres or not in interaction with the system B of N particles. But, given the symmetry of the whole system, the same arguments can be used to decide whether the system B of N particles decoheres or not in interaction with the system A of M particles, with analogous results: B decoheres only when $M \gg N$; if $M \ll N$ or $M \simeq N$, B does not decohere. Therefore, all the results obtained in this section can be summarized as follows.

(i) If $M \ll N$, A decoheres and B does not decohere.
(ii) If $M \gg N$, A does not decohere and B decoheres.
(iii) If $M \simeq N$, neither A nor B decohere.

5. Generalized spin-bath model: decomposition 2

5.1. Selecting the relevant observables

In this case we decide to observe only one particle of the open system A. This amounts to splitting the closed system U into two new subsystems: the open system S is, say, the particle A_M with ket $|\Uparrow, \Uparrow, \ldots, \Uparrow, \Uparrow, \Uparrow, \Downarrow\rangle$, and the environment is $E = \left(\cup_{i=1}^{M-1} A_i\right) \cup B = \left(\cup_{i=1}^{M-1} A_i\right) \cup \left(\cup_{k=1}^{N} B_k\right)$. The decomposition for this case is

$$\mathcal{H} = \mathcal{H}_S \otimes \mathcal{H}_E = \left(\mathcal{H}_{A_M}\right) \otimes \left(\left(\bigotimes_{i=1}^{M-1} \mathcal{H}_{A_i}\right) \otimes \left(\bigotimes_{k=1}^{N} \mathcal{H}_{B_k}\right)\right). \tag{69}$$

Therefore, the relevant observables O_R of the closed system U are those corresponding to the particle A_M:

$$O_R = O_S \otimes \mathbb{I}_E = \left(\sum_{\alpha,\alpha'=\Uparrow,\Downarrow} s_{\alpha,\alpha'}\, |\alpha\rangle\langle\alpha'|\right) \otimes \left(\left(\bigotimes_{i=1}^{M-1} \mathbb{I}_i\right) \otimes \left(\bigotimes_{k=1}^{N} \mathbb{I}_k\right)\right). \tag{70}$$

It is easy to see that the relevant observables selected in this decomposition 2 form a subspace of the space of the relevant observables selected in decomposition 1: equation (70) can be obtained from equation (52) by making $s_{\lambda,\lambda'} = 1$ for $\lambda = \lambda'$ and $s_{\lambda,\lambda'} = 0$ for $\lambda \neq \lambda'$ in all the terms of the sum except for the terms corresponding to the particle A_M.

In order to simplify expressions, in this case it is convenient to introduce a new arrangement for the eigenvectors of the Hamiltonian H_A, by calling them $|\tilde{\mathcal{A}}_i\rangle$: the set $\{|\tilde{\mathcal{A}}_i\rangle\}$ is an eigenbasis of H_A with 2^M elements. The $|\tilde{\mathcal{A}}_i\rangle$ will be ordered by analogy with the binary numbers:

$$\begin{aligned}
&|\tilde{\mathcal{A}}_1\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Uparrow, \Uparrow, \Uparrow\rangle, |\tilde{\mathcal{A}}_2\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Uparrow, \Uparrow, \Downarrow\rangle,\\
&|\tilde{\mathcal{A}}_3\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Uparrow, \Downarrow, \Uparrow\rangle, |\tilde{\mathcal{A}}_4\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Uparrow, \Downarrow, \Downarrow\rangle,\\
&|\tilde{\mathcal{A}}_5\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Downarrow, \Uparrow, \Uparrow\rangle, |\tilde{\mathcal{A}}_6\rangle = |\Uparrow, \Uparrow, \ldots, \Uparrow, \Downarrow, \Uparrow, \Downarrow\rangle, \ldots\\
&|\tilde{\mathcal{A}}_{2^M}\rangle = |\Downarrow, \Downarrow, \ldots, \Downarrow, \Downarrow, \Downarrow, \Downarrow\rangle.
\end{aligned} \tag{71}$$

According to this arrangement, the $|\tilde{\mathcal{A}}_i\rangle$ with even i have the spin M in the state $|\Downarrow\rangle$, and the $|\tilde{\mathcal{A}}_i\rangle$ with odd i have the spin M in the state $|\Uparrow\rangle$. So, the relevant observables of equation (70) can be rewritten in terms of the $|\tilde{\mathcal{A}}_i\rangle$ as

$$O_R = \left(\sum_{\lambda=1}^{2^M} (\tilde{s}_{\Uparrow\Uparrow}|\tilde{\mathcal{A}}_{2\lambda}\rangle\langle\tilde{\mathcal{A}}_{2\lambda}| + \tilde{s}_{\Uparrow\Downarrow}|\tilde{\mathcal{A}}_{2\lambda}\rangle\langle\tilde{\mathcal{A}}_{2\lambda-1}| + \tilde{s}_{\Downarrow\Uparrow}|\tilde{\mathcal{A}}_{2\lambda-1}\rangle\langle\tilde{\mathcal{A}}_{2\lambda}| + \tilde{s}_{\Downarrow\Downarrow}|\tilde{\mathcal{A}}_{2\lambda-1}\rangle\langle\tilde{\mathcal{A}}_{2\lambda-1}|)\right) \otimes \left(\bigotimes_{k=1}^{N} \mathbb{I}_k\right). \tag{72}$$

5.2. Computing the behavior of the relevant expectation values

Here the expectation values of the relevant observables are given by equation (47), with $T_{l,l'}(t)$, $T_{l,l}(t)$ and $T_{l,M-l}(t)$ given by equations (45), (49) and (50) respectively, but now replacing $B_{\lambda,\lambda'}$ with $\tilde{B}_{\lambda,\lambda'}$,

$$\langle O_R\rangle_{\rho(t)} = \sum_{l=0}^{M}\sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(l-1)+1}}^{f(l)} \tilde{B}_{\lambda,\lambda'} + \sum_{l=0}^{\tilde{M}}\sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(M-l-1)+1}}^{\substack{f(l)\\ f(M-l)}} \tilde{B}_{\lambda,\lambda'} 2\,\mathrm{Re}(T_{l,M-l}(t))$$

$$+ \sum_{\substack{l,l'=0\\ l\neq l'\\ l'\neq M-l}}^{M}\sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(l'-1)+1}}^{\substack{f(l)\\ f(l')}} \tilde{B}_{\lambda,\lambda'} T_{l,l'}(t), \tag{73}$$

where the $\tilde{B}_{\lambda,\lambda'}$ can be written in the basis $\{|\tilde{\mathcal{A}}_\lambda\rangle\}$ as

$$\tilde{B}_{\lambda,\lambda'} = \begin{cases} C_\lambda C^*_{\lambda'}\tilde{s}_{\Uparrow\Uparrow} & \text{if} \quad \lambda \quad \text{is an even number and} \quad \lambda'=\lambda \\ C_\lambda C^*_{\lambda'}\tilde{s}_{\Uparrow\Downarrow} & \text{if} \quad \lambda \quad \text{is an even number and} \quad \lambda'=\lambda-1 \\ C_\lambda C^*_{\lambda'}\tilde{s}_{\Downarrow\Uparrow} & \text{if} \quad \lambda \quad \text{is an odd number and} \quad \lambda'=\lambda+1 \\ C_\lambda C^*_{\lambda'}\tilde{s}_{\Downarrow\Downarrow} & \text{if} \quad \lambda \quad \text{is an odd number and} \quad \lambda'=\lambda \\ 0 & \qquad\qquad \text{otherwise} \end{cases}. \tag{74}$$

According to equation (74), $\tilde{B}_{\lambda,\lambda'} \neq 0$ only when

$$\lambda'=\lambda \qquad \text{or} \qquad \lambda'=\lambda\pm 1. \tag{75}$$

Since $\lambda = f(l-1)+1$ and $\lambda' = f(l'-1)+1$, relations (75) imply that

$$l'=l \qquad \text{or} \qquad l'=l\pm 1. \tag{76}$$

The expectation value given by equation (73) has again three terms, $\langle O\rangle_{\rho(t)} = \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}$, which can be analyzed separately.

- From equations (74) and (75), the first term reads

$$\Sigma^{(1)} = \sum_{l=0}^{M}\sum_{\lambda=f(l-1)+1}^{f(l)} B_{\lambda,\lambda} = \sum_{\lambda=0}^{2^{M-1}}\left(|C_{2\lambda}|^2\,\tilde{s}_{\Uparrow\Uparrow} + |C_{2\lambda+1}|^2\,\tilde{s}_{\Downarrow\Downarrow}\right) \neq \Sigma^{(1)}(t). \tag{77}$$

  Analogously to equation (58) of decomposition 1, this first term does not evolve with time.
- The time dependence of the second term is given by $T_{l,M-l}(t)$. But with the restrictions of equations (75) and (76), $\Sigma^{(2)}$ has only two terms:

$$\Sigma^{(2)}(t) = \sum_{l=0}^{\tilde{M}}\sum_{\substack{\lambda=f(l-1)+1\\ \lambda'=f(M-l-1)+1}}^{\substack{f(l)\\ f(M-l)}} B_{\lambda,\lambda'} 2\,\mathrm{Re}(T_{l,M-l}(t)) \tag{78}$$

$$= C_{f(\frac{M-1}{2}-1)+1}C^*_{f(\frac{M-1}{2}-1)+2}(\tilde{s}_{\Downarrow\Uparrow} + \tilde{s}_{\Uparrow\Downarrow})2\,\mathrm{Re}\big(T_{\frac{M-1}{2},\frac{M+1}{2}}(t)\big). \tag{79}$$

  Then, in order to obtain the limit of this term, we have to compute the limit of $T_{\frac{M-1}{2},\frac{M+1}{2}}(t)$, which is precisely the $T_{l,l'}(t)$ of decomposition 1 in the particular case that $l = \frac{M-1}{2}$ and $l' = \frac{M+1}{2}$ (see equation (55)). But, as we have seen in case (a) of decomposition 1, $T_{l,l'}(t)$ has the same functional form as $\Gamma_1(t)$ of the original model (see equation (10)), which approaches zero for $t \to \infty$ when $N \gg 1$. Therefore, for $N \gg 1$, $T_{\frac{M-1}{2},\frac{M+1}{2}}(t)$ also approaches zero for $t \to \infty$, and the same holds for $\Sigma^{(2)}(t)$ since it is a sum of two terms containing $T_{\frac{M-1}{2},\frac{M+1}{2}}(t)$.

- The time dependence of the third term is given by $T_{l,l'}(t)$. But with the restrictions of equations (75) and (76), $\Sigma^{(3)}$ results

$$\Sigma^{(3)}(t) = \sum_{\substack{l=0 \\ l\neq \frac{M-1}{2}}}^{M} \sum_{\lambda=f(l-1)+1}^{f(l)} (B_{\lambda,\lambda+1} T_{l,l+1}(t) + B_{\lambda,\lambda-1} T_{l,l-1}(t)). \tag{80}$$

Since here $l' = l \pm 1$ (see equation (76)), in this case $T_{l,l\pm 1}(t)$ is

$$T_{l,l\pm 1}(t) = \prod_{j=1}^{N} (|\alpha_j|^2 \, \mathrm{e}^{\mp i g_j t} + |\beta_j|^2 \, \mathrm{e}^{\pm i g_j t}). \tag{81}$$

If we compare this equation with equation (14) for $r(t)$ in the original spin-bath model, we can see that

$$T_{l,l+1}(t) = r(t) \qquad \text{and} \qquad T_{l,l-1}(t) = r^*(t). \tag{82}$$

Then,

$$\Sigma^{(3)}(t) = (S_+ r(t) + S_- r^*(t)), \tag{83}$$

where S_+ and S_- are constants given by

$$S_\pm = \sum_{\substack{l=0 \\ l\neq \frac{M-1}{2}}}^{M} \sum_{\lambda=f(l-1)+1}^{f(l)} B_{\lambda,\lambda\pm 1}. \tag{84}$$

On the basis of the simulations of the original model we have seen that, when $N \gg 1$, $r(t)$ approaches zero for $t \to \infty$. Therefore, in this case we can conclude that, when $N \gg 1$, $\Sigma^{(3)}(t)$ approaches zero for $t \to \infty$.

Summing up, $\langle O_R \rangle_{\rho(t)}$ is the sum of three terms: one is time independent and the other two tend to zero for $t \to \infty$. In particular, from equation (77) we know that, for $N \gg 1$,

$$\lim_{t\to\infty} \langle O_R \rangle_{\rho(t)} = \sum_{l=0}^{M} \sum_{\lambda,\lambda'=f(l-1)+1}^{f(l)} \tilde{B}_{\lambda,\lambda'} = \sum_{l=0}^{M} \sum_{\lambda=0}^{2^{M-1}} \left(|C_{2\lambda}|^2 \, \tilde{s}_{\Uparrow\Uparrow} + |C_{2\lambda+1}|^2 \, \tilde{s}_{\Downarrow\Downarrow}\right) = \langle O_R \rangle_{\rho_*}, \tag{85}$$

where ρ_* is the final diagonal state of U. Again, this result can also be expressed in terms of the reduced density operator $\rho_S = \rho_{A_M}$ of the open system $S = A_M$ as (see equation (65)):

$$\lim_{t\to\infty} \langle O_R \rangle_{\rho(t)} = \langle O_R \rangle_{\rho_*} = \lim_{t\to\infty} \langle O_{A_M} \rangle_{\rho_{A_M}(t)} = \langle O_{A_M} \rangle_{\rho_{A_M *}}, \tag{86}$$

where the final reduced density operator $\rho_{A_M *}$ in the basis $\{|\Uparrow\rangle, |\Downarrow\rangle\}$ reads

$$\rho_{A_M *} = \begin{pmatrix} |\alpha_M|^2 & 0 \\ 0 & |\beta_M|^2 \end{pmatrix}. \tag{87}$$

This shows that the open system $S = A_M$, composed of a single particle, decoheres in interaction with its environment E of $N + M - 1$ particles when $N \gg 1$, independently of the value of M.

In order to illustrate this conclusion, we have computed $\Sigma^{nd}(t) = \Sigma^{(2)}(t) + \Sigma^{(3)}(t)$ by means of numerical simulations with the same features as in decomposition 1, with the exception of condition (vi), which was taken as:

Figure 5: (vi) $M = 10^3$ and $N = 1$.
Figure 6: (vi) $M = 10^3$ and $N = 10^2$.
Figure 7: (vi) $M = 10^3$ and $N = 10^3$.

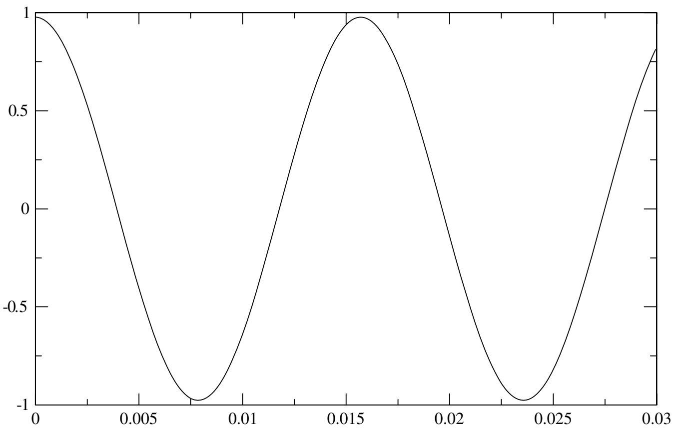

Figure 5. Evolution of $\Sigma^{nd}(t)$ for $M = 10^3$ and $N = 1$, with $t_0 = 3.10^{-2}$s.

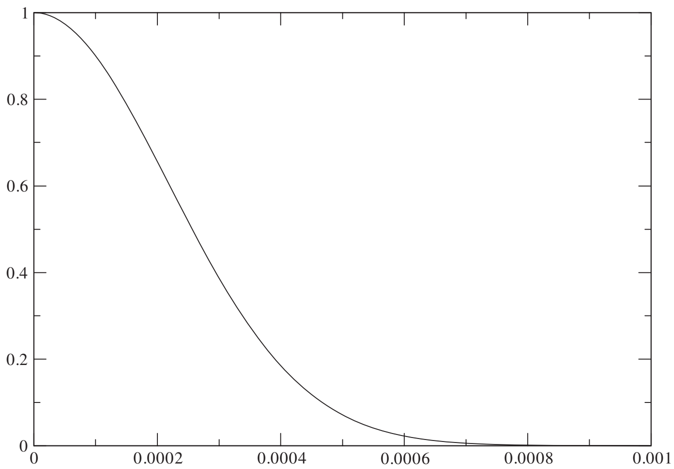

Figure 6. Evolution of $\Sigma^{nd}(t)$ for $M = 10^3$ and $N = 10^2$, with $t_0 = 1.10^{-3}$ s.

Summarizing results

As we have seen, in this decomposition of the whole closed system, the open system $S = A_M$ decoheres when $N \gg 1$, independently of the value of M. But the particle A_M was selected as S only for computation simplicity: the same argument can be developed for any particle A_i of A. Then, when $N \gg 1$ and independently of the value of M, any particle A_i decoheres in interaction with its environment E of $N + M - 1$ particles.

On the other hand, as in decomposition 1, here the symmetry of the whole system U allows us to draw analogous conclusions when the system S is one of the particles of B, say, B_N: $S = B_N$ decoheres when $M \gg 1$, independently of the value of N. And, on the basis of the same considerations as above, when $M \gg 1$ and independently of the value of N, any particle B_i decoheres in interaction with its environment E of $N + M - 1$ particles.

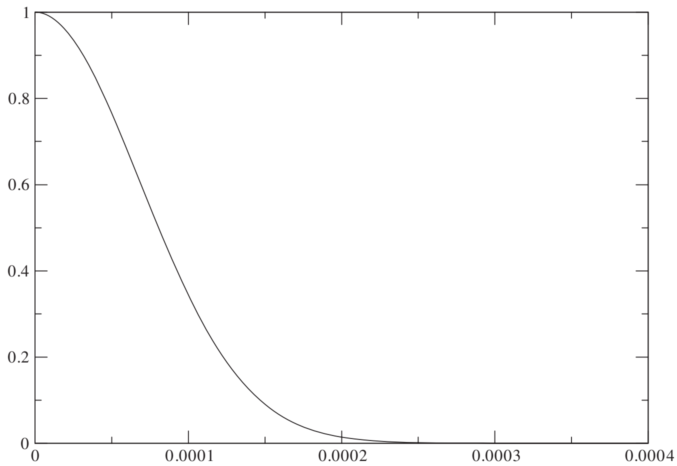

Figure 7. Evolution of $\Sigma^{nd}(t)$ for $M = 10^3$ and $N = 10^3$, with $t_0 = 4.10^{-4}$ s.

6. Concluding remarks

In this paper we have studied a generalization of the spin-bath model, where a closed system U is composed by two subsystems, $U = A \cup B$, with A of M particles A_i and B of N particles B_i. We showed how the model behaves under different definitions of the system of interest and under different relations between the numbers M and N. The results obtained here allow us to state the following concluding remarks.

(a) We have seen that, when $M \gg N$ or $M \simeq N$, the subsystem A does not decohere (decomposition 1 of section 4), but the particles A_i, considered independently, decohere when $N \gg 1$ (decomposition 2 of section 5). This means that there are physically meaningful situations, given by $M \gg N \gg 1$ or $M \simeq N \gg 1$, where all the A_i decohere although A does not decohere. In other words, in spite of the fact that certain particles decohere and may behave classically, the subsystem composed by all of them retains its quantum nature. We have also seen that, by symmetry, all the particles B_i, considered independently, also decohere when $M \gg 1$. Then, when $M \gg N \gg 1$ or $M \simeq N \gg 1$, the requirement $M \gg 1$ holds and we can conclude that not only all the A_i, but also all the B_i decohere, although B neither decoheres. So, all the particles of the closed system $U = (\cup_i A_i) \cup (\cup_j B_j)$ may become classical when considered independently, although the whole system U certainly does not decohere and, therefore, retains its quantum character. These results, considered together, are a clear manifestation of the fact, already pointed out by Schlosshauer [17], that energy dissipation and decoherence are different phenomena: since all the particles of the system U decohere when independently considered, decoherence cannot result from the dissipation of energy from the decohered systems to their environments.

(b) The generalized model shows that the split of the entire closed system into an open system and its environment amounts to the selection of the observables relevant in each situation. Since there is no privileged or essential decomposition, we can select the observables of the subsystem A in the situation in which A does not decohere. In this way, it would be possible to use appropriately selected subsystems, unaffected by decoherence, for storing quantum information.

(c) The natural further step of generalization will consist in following the ideas of paper [6], and introducing coupling internal to the subsystems A or B. For instance, given that the decoherence of A is increasingly suppressed as the number M of its particles increases, it could be expected that such decoherence suppression will also be more efficient as the interactions between the spins of the bath also increase.

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